# Covariance function of vector self-similar process

Frédéric Lavancier, Anne Philippe, Donatas Surgailis

# Covariance function of vector self-similar process *

Frédéric Lavancier[(1)], Anne Philippe[(1)], Donatas Surgailis[(2)†]

August 25, 2009

**1.** Laboratoire de Mathématiques Jean Leray, Université de Nantes, France

{Frederic.Lavancier,Anne.Philippe}@univ-nantes.fr

**2.** Institute of Mathematics and Informatics, Vilnius, Lithuania

sdonatas@ktl.mii.lt

**Abstract**

The paper obtains the general form of the cross-covariance function of vector fractional Brownian motion with correlated components having different self-similarity indices.

*Keywords:* operator self-similar process, vector fractional Brownian motion, cross-covariance function.

## 1 Introduction

A $p-$variate stochastic process $X = \{X(t) = (X_1(t), \cdots, X_p(t)), t \in \mathbb{R}\}$ is said *operator self-similar* (os-s) if there exists a $p \times p$ matrix $H$ (called the exponent of $X$) such that for any $\lambda > 0$,

$$X(\lambda t) \quad =_{\mathrm{fdd}} \quad \lambda^H X(t), \tag{1.1}$$

where $=_{\mathrm{fdd}}$ means equality of finite-dimensional distributions, and the $p \times p$ matrix $\lambda^H$ is defined by the power series $\lambda^H = \mathrm{e}^{H \log \lambda} = \sum_{k=0}^{\infty} H^k (\log \lambda)^k / k!$. Os-s processes were studied in Laha and Rohatgi (1981), Hudson and Mason (1982), Maejima and Hudson (1984), Sato (1991) and other papers. A Gaussian os-s process with stationary increments

*The joint work was partially supported by the research project MATPYL of the Fédération de Mathématiques des Pays de Loire.

†The author is partially supported by the Lithuanian State Science Foundation grant no. T-70/09.

is called operator fractional Brownian motion (ofBm). For $p = 1$ the class of ofBm coincides with fundamental class of fractional Brownian motions (fBm) (see e.g. Samorodnitsky and Taqqu (1994)). Recall that a fBm with exponent $H \in (0, 1)$ can be alternatively defined as a stochastically continuous Gaussian process $X = \{X(t), t \in \mathbb{R}\}$ with zero mean and covariance

$$\mathrm{E}X(s)X(t) = \frac{\sigma^2}{2}(|s|^{2H} + |t|^{2H} - |t-s|^{2H}), \qquad t, s \in \mathbb{R}, \tag{1.2}$$

where $\sigma^2 = \mathrm{E}X^2(1)$. The form of covariance of general ofBm seems to be unknown and may be quite complicated. The structure of ofBm and stochastic integral representations were studied in Didier and Pipiras (2008).

A particular case of os-s processes corresponds to *diagonal* matrix $H = \mathrm{diag}(H_1, \cdots, H_p)$. In this case, relation (1.1) becomes

$$(X_1(\lambda t), \cdots, X_p(\lambda t)) \quad =_{\mathrm{fdd}} \quad (\lambda^{H_1} X_1(t), \cdots, \lambda^{H_p} X_p(t)). \tag{1.3}$$

Below, a $p-$variate process $X$ satisfying (1.3) for any $\lambda > 0$ will be called *vector self-similar* (vs-s) and a stochastically continuous Gaussian vs-s process with stationary increments (si) will be called a *vector fractional Brownian motion* (vfBm).

Vs-s processes (in particularly, vfBm) seem to be most useful for applications and statistical analysis of multiple time series. They arise as limits of normalized partial sums of multivariate long memory processes with discrete time, see Marinucci and Robinson (2000), Davidson and de Jong (2000), Chung (2002), Davidson and Hashimadze (2008), Robinson (2008), Lavancier *et al.* (2009). In particularly, vs-s processes appear in the OLS estimation in multiple linear regression model (Chung, 2002), the multiple local Whittle estimation (Robinson, 2008) and the two-sample testing for comparison of long memory parameters (Lavancier *et al.*, 2009).

Note from (1.3) that each component $X_i = \{X_i(t), t \in \mathbb{R}\}$, $i = 1, \cdots, p$ of vs-s process is a (scalar) self-similar process, the fact which is not true for general os-s processes.

The present paper obtains the general form of the (cross-)covariance function of vs-s si process $X$ with finite variance and exponent $H = \mathrm{diag}(H_1, \cdots, H_p)$, $0 < H_i < 1$. According to Theorem 2.1 below, under some regularity condition, for any $i, j = 1, \cdots, p, i \neq j$ with $H_i + H_j \neq 1$, there exist $c_{ij}, c_{ji} \in \mathbb{R}$ such that for any $s, t \in \mathbb{R}$

$$\mathrm{cov}(X_i(s), X_j(t)) \;=\; \frac{\sigma_i \sigma_j}{2} \left\{ c_{ij}(s)|s|^{H_i+H_j} + c_{ji}(t)|t|^{H_i+H_j} - c_{ji}(t-s)|t-s|^{H_i+H_j} \right\}, \tag{1.4}$$

where $\sigma_i^2 := \mathrm{var}(X_i(1))$ and

$$c_{ij}(t) \;:=\; \begin{cases} c_{ij}, & t > 0, \\ c_{ji}, & t < 0. \end{cases} \tag{1.5}$$

A similar expression (involving additional logarithmic terms) for the covariance $\mathrm{cov}(X_i(s),$ $X_j(t))$ is obtained in the case $H_i + H_j = 1$. We prove Theorem 2.1 by deriving from vs-s si property a functional equation for the cross-covariance of the type studied in Aczél

(1966) and Aczél and Hosszú (1965) and using the result of the last paper (Theorem 2.2 below) about the uniqueness of this equation. Section 3 discusses the existence of vfBm with covariance as in (1.4). We start with a double sided stochastic integral representation similar to Didier and Pipiras (2008):

$$X(t) = \int_{\mathbb{R}} \left\{ \left( (t-x)_+^{H-.5} - (-x)_+^{H-.5} \right) A_+ + \left( (t-x)_-^{H-.5} - (-x)_-^{H-.5} \right) A_- \right\} W(\mathrm{d}x), \tag{1.6}$$

where $H - .5 := \mathrm{diag}(H_1 - .5, \cdots, H_p - .5), x_+ := \max(x, 0), x_- := \max(-x, 0), A_+, A_-$ are real $p \times p$ matrices and $W(\mathrm{d}x) = (W_1(\mathrm{d}x), \cdots, W_p(\mathrm{d}x))$ is a Gaussian white noise with zero mean, independent components and covariance $\mathrm{E}W_i(\mathrm{d}x)W_j(\mathrm{d}x) = \delta_{ij}\mathrm{d}x$. According to Proposition 3.1, if $0 < H_i < 1, H_i + H_j \neq 1,\ i, j = 1, \cdots, p$ then the cross-covariance of $X$ in (1.6) has the form as in (1.4) with

$$c_{ij} \ = \ 2\tilde{c}_{ij}\phi_{ij}/\sigma_i\sigma_j, \quad \phi_{ij} := B(H_i + .5, H_j + .5)/\sin((H_i + H_j)\pi), \tag{1.7}$$

where the matrix $\tilde{C} = (\tilde{c}_{ij})$ is given by

$$\begin{aligned} \tilde{C} \ &:= \ \cos(H\pi)A_+A_+^* + A_-A_-^* \cos(H\pi) \\ &- \ \sin(H\pi)A_+A_-^* \cos(H\pi) - \cos(H\pi)A_+A_-^* \sin(H\pi). \end{aligned} \tag{1.8}$$

Here and below, $A^*$ denotes the transposed matrix, $\sin(H\pi) := \mathrm{diag}(\sin(H_1\pi), \cdots, \sin(H_p\pi))$, $\cos(H\pi) := \mathrm{diag}(\cos(H_1\pi), \cdots, \cos(H_p\pi))$.

## 2 The form of the covariance function of vs-s process

Recall that a random process $X = \{X(t), t \in \mathbb{R}\}$ has stationary increments (si) if $\{X(t+T) - X(T), t \in \mathbb{R}\} \ =_{\mathrm{fdd}} \ \{X(t) - X(0), t \in \mathbb{R}\}$ for any $T \in \mathbb{R}$.

**Theorem 2.1** *Let $X = \{X(t), t \in \mathbb{R}\}$ be a 2nd order process with values in $\mathbb{R}^p$. Assume that $X$ has stationary increments, zero mean, $X(0) = 0$, and that $X$ is vector self-similar with exponent $H = \mathrm{diag}(H_1, \cdots, H_p),\ 0 < H_i < 1\ (i = 1, \cdots, p)$.*

*Moreover, assume also that for any $i, j = 1, \cdots, p$, the function $t \mapsto \mathrm{E}X_i(t)X_j(1)$ is continuously differentiable on $(0, 1) \cup (1, \infty)$. Let $\sigma_i^2 > 0$ denote the variance of $X_i(1),\ i = 1, \cdots, p$.*

*(i) If $i = j$, then for any $(s, t) \in \mathbb{R}^2$, we have*

$$\mathrm{E}X_i(s)X_i(t) \ = \frac{\sigma_i^2}{2} \left\{ |s|^{2H_i} + |t|^{2H_i} - |t-s|^{2H_i} \right\}. \tag{2.1}$$

*(ii) If $i \neq j$ and $H_i + H_j \neq 1$, then there exists $c_{ij}, c_{ji} \in \mathbb{R}$ such that for any $(s, t) \in \mathbb{R}^2$, (1.4) holds.*

*(iii) If $i \neq j$ and $H_i + H_j = 1$, then there exists $d_{ij}, f_{ij} \in \mathbb{R}$ such that for any $(s,t) \in \mathbb{R}^2$, we have*

$$\mathrm{E}X_i(s)X_j(t) = \frac{\sigma_i\sigma_j}{2}\{d_{ij}(|s|+|t|-|s-t|) + f_{ij}(t\log|t| - s\log|s| - (t-s)\log|t-s|)\}. \tag{2.2}$$

*(iv) The matrix $R = (R_{ij})_{i,j=1,\cdots,p}$ is positive definite, where*

$$R_{ij} := \begin{cases} 1, & i = j, \\ c_{ij} + c_{ji}, & i \neq j, \ H_i + H_j \neq 1, \\ d_{ij}, & i \neq j, \ H_i + H_j = 1. \end{cases}$$

*Proof.* (i) Follows from the well-known characterization of covariance of (scalar-valued) self-similar stationary increment process.

(ii)-(iii) From part (i), it follows that $\mathrm{E}(X_i(t) - X_i(s))^2 = \sigma_i^2|t-s|^{2H_i}$ and hence $\{X_i(t), t \in \mathbb{R}\}$ is stochastically continuous on the real line, for any $i = 1, \cdots, p$. Whence, it follows that $\mathrm{E}X_i(s)X_j(t)$ is jointly continuous on $\mathbb{R}^2$ and vanishes for $t = 0$ or $s = 0$, for any $i, j = 1, \cdots, p$.

Denote $r(s,t) := \mathrm{E}X_i(s)X_j(t)$, $H := \frac{1}{2}(H_i + H_j)$. Let $\mathbb{R}_+ := \{u : u > 0\}$, $\mathbb{R}_- := \{u : u < 0\}$. From vs-s and si properties we obtain

$$r(\lambda s, \lambda t) = \lambda^{2H} r(s,t), \tag{2.3}$$

$$r(s,t) = r(s+T, t+T) - r(s+T, T) - r(T, t+T) + r(T,T) \tag{2.4}$$

for any reals $s, t, T$ and any $\lambda > 0$. Substituting $s = t = u, \lambda = 1/|u|$ into (2.3) one obtains

$$r(u,u) = 2\kappa_\pm|u|^{2H}, \quad u \in \mathbb{R}_\pm, \quad 2\kappa_\pm := r(\pm 1, \pm 1). \tag{2.5}$$

Substituting $s = t = -1, T = 1$ into (2.4) and using $r(0,0) = r(1,0) = r(0,1) = 0$ yields $r(-1,-1) = r(1,1)$. Next, substituting $T = u, s = t = v - u$ into (2.4) and using (2.5) with $\kappa := \kappa_+ = \kappa_-$ one obtains

$$\begin{aligned} r(u,v) + r(v,u) &= r(v,v) + r(u,u) - r(v-u, v-u) \\ &= 2\kappa(|v|^{2H} + |u|^{2H} - |u-v|^{2H}) \qquad (u, v \in \mathbb{R}). \end{aligned} \tag{2.6}$$

Next, let

$$g_\pm(t) := r(\pm 1, t) \qquad (t \in \mathbb{R}).$$

Then (2.3) implies

$$r(s,t) = |s|^{2H} g_\pm(\pm t/s) \qquad (s \in \mathbb{R}_\pm, \ t \in \mathbb{R}). \tag{2.7}$$

Equation (2.4) with $s = 1$ and (2.7) yield, for all $t \in \mathbb{R}$

$$\begin{aligned} g_+(t) &= (T+1)^{2H} g_+\left(\frac{T+t}{T+1}\right) - T^{2H} g_+\left(\frac{T+t}{T}\right) \\ &- (T+1)^{2H} g_+\left(\frac{T}{T+1}\right) + T^{2H} g_+(1) \qquad (T > 0), \end{aligned} \tag{2.8}$$

and for all $t \in \mathbb{R}$, equation (2.4) with $s = -1$ and (2.7) yield

$$
\begin{aligned}
g_-(t) &= (T-1)^{2H} g_+\left(\frac{T+t}{T-1}\right) - T^{2H} g_+\left(\frac{T+t}{T}\right) \\
&- (T-1)^{2H} g_+\left(\frac{T}{T-1}\right) + T^{2H} g_+(1) \qquad (T>1).
\end{aligned} \tag{2.9}
$$

We claim that the general solution of functional equations (2.8)-(2.9) has the form:

- When $H \neq 1/2$,

$$
g_+(t) = \begin{cases} c' + c'|t|^{2H} - c'|1-t|^{2H}, & t<0, \\ c' + c|t|^{2H} - c'|1-t|^{2H}, & 0<t<1, \\ c' + c|t|^{2H} - c|1-t|^{2H}, & t>1, \end{cases} , \tag{2.10}
$$

$$
g_-(t) = \begin{cases} c + c'|t|^{2H} - c'|t+1|^{2H}, & t<-1, \\ c + c'|t|^{2H} - c|t+1|^{2H}, & -1<t<0, \\ c + c|t|^{2H} - c|t+1|^{2H}, & t>0; \end{cases} \tag{2.11}
$$

- When $H = 1/2$,

$$
g_+(t) = \begin{cases} f\left(t\log|t| - (t-1)\log|t-1|\right), & t<0, \\ d\,(1\wedge t) + f\left(t\log|t| - (t-1)\log|t-1|\right), & t>0, \end{cases} , \tag{2.12}
$$

$$
g_-(t) = \begin{cases} d\,(1\wedge |t|) + f\left(t\log|t| - (t+1)\log|t+1|\right), & t<0, \\ f\left(t\log|t| - (t+1)\log|t+1|\right), & t>0, \end{cases} \tag{2.13}
$$

with some $c, c', d, f \in \mathbb{R}$.

It follows from (2.10)-(2.13) and (2.7) that the covariance $r(s,t) = \mathrm{E}X_i(s)X_j(t)$ for $(s,t) \in \mathbb{R}^2$ has the form as in (1.4), (2.2), with $c_{ij} = 2c/\sigma_i\sigma_j, c_{ji} = 2c'/\sigma_i\sigma_j$ in the case $2H = H_i + H_j \neq 1$, and $d_{ij} = 2d/\sigma_i\sigma_j,\ f_{ij} = 2f/\sigma_i\sigma_j$ in the case $2H = H_i + H_j = 1$.

To show the above claim, note, by direct verification, that (2.10)-(2.13) solve equations (2.8)-(2.9). Therefore it suffices to show that (2.10)-(2.13) is a unique solution of (2.8)-(2.9).

Let $t > 1$. Differentiating (2.8) with respect to $t$ leads to

$$
g'_+(t) = (T+1)^{2H-1} g'_+\Big(\frac{T+t}{T+1}\Big) - T^{2H-1} g'_+\Big(\frac{T+t}{T}\Big). \tag{2.14}
$$

Let $x := t,\ y := (T+t)/T$. Then $(x,y) \in (1,\infty)^2$ and the mapping $(t,T) \mapsto (x,y) : (1,\infty)\times(0,\infty) \to (1,\infty)^2$ is a bijection. Equation (2.14) can be rewritten as

$$
g'_+(F(x,y)) = \Big(\frac{y-1}{x+y-1}\Big)^{2H-1} g'_+(x) + \Big(\frac{x}{x+y-1}\Big)^{2H-1} g'_+(y), \tag{2.15}
$$

where

$$
F(x,y) := \frac{xy}{x+y-1}. \tag{2.16}
$$

Equation (2.15) belongs to the class of functional equations treated in Aczél (1966) and Aczél and Hosszú (1965). For reader's convenience, we present the result from Aczél (1966) which will be used below.

**Theorem 2.2** (Aczél and Hosszú (1965)) *There exists at most one continuous function $f$ satisfying the functional equation*

$$f(F(x,y)) \;=\; L(f(x), f(y), x, y) \tag{2.17}$$

*for all $x, y \in \langle A, B\rangle$ and the initial conditions*

$$f(a_1) = b_1, \quad f(a_2) = b_2 \qquad (a_1, a_2 \in \langle A, B\rangle,\; a_1 \neq a_2)$$

*if $F$ is continuous in $\langle A, B\rangle \times \langle A, B\rangle$, $L(u, v, x, y)$ is strictly monotonic in $u$ or $v$ and $F$ is intern (i.e., $F(x,y) \in (x,y)$ for all $x \neq y \in \langle A, B\rangle$), where $\langle A, B\rangle$ is a closed, half-closed, open, finite or infinite interval.*

Unfortunately, equation (2.15) does not satisfy the conditions of Theorem 2.2 (with $f = g_+$ and $\langle A, B\rangle = (1, \infty)$), since $F$ in (2.16) is not intern. Following Aczél and Hosszú (1965), we first apply some transformations of (2.15) so that Theorem 2.2 can be used.

Note, taking $T = t/(t-1)$ in (2.14) one obtains

$$g'_+(t) \;=\; K(t) g'_+\Big(\frac{t^2}{2t-1}\Big), \tag{2.18}$$

where

$$K(t) \;:=\; \frac{(2t-1)^{2H-1}}{t^{2H-1} + (t-1)^{2H-1}}. \tag{2.19}$$

Let $\tilde{F}(x) := F(x,x) = \frac{x^2}{2x-1}$. Then $\tilde{F}$ is strictly increasing from $(1, \infty)$ onto $(1, \infty)$. Let

$$G(x,y) \;:=\; \tilde{F}^{-1}(F(x,y)) \;=\; \frac{xy + \sqrt{x(x-1)y(y-1)}}{x+y-1}, \tag{2.20}$$

with $\tilde{F}^{-1}(y) = y + \sqrt{y(y-1)}$. Note $F(x,y) > 1,\; G(x,y) > 1$ for $(x,y) \in (1,\infty)^2$. From (2.18) with $t = G(x,y)$ one obtains

$$g'_+(G(x,y)) \;=\; g'_+(F(x,y)) K(G(x,y)). \tag{2.21}$$

Combining (2.15) and (2.21) one gets

$$g'_+(G(x,y)) \;=\; L(g'_+(x), g'_+(y), x, y), \tag{2.22}$$

where

$$L(u,v,x,y) \;:=\; \left(\Big(\frac{y-1}{x+y-1}\Big)^{2H-1} u + \Big(\frac{x}{x+y-1}\Big)^{2H-1} v\right) K(G(x,y)). \tag{2.23}$$

The fact that $G$ in (2.20) is intern follows from its definition and monotonicity in $x$ and $y$, implying

$$x = G(x,x) \leq G(x,y) \leq G(y,y) = y$$

for any $x \le y,\ x, y \in (1, \infty)$. Since $L$ is monotonic in $u$ or $v$, so Theorem 2.2 applies to functional equation (2.22) and therefore this equation has a unique continuous solution $g'_+$ on the interval $(1, \infty)$, given boundary conditions $g'_+(a_i) = b_i\ (i = 1, 2),\ 1 < a_2 < a_1 < \infty$.

Form of $g_+$ on $[1, +\infty)$ :

- Assume $H \neq 1/2$. Let $a_1 := 2, b_1 := cH(2^{2H} - 2)$. In view of (2.18), the other boundary condition can be defined by $a_2 = a_1^2/(2a_1 - 1) = 4/3,\ b_2 := g'_+(2)/K(2)$ and so equations (2.22) and (2.14) have a unique solution for single boundary condition $g'_+(2) = cH(2^{2H} - 2)$. (See also Aczél and Hosszú (1965, p.51).) Since $g'_+(t) = 2Hc(t^{2H-1} - (t-1)^{2H-1})$ is a solution of (2.14) with this boundary condition, it follows that this solution is unique. Hence it also follows that

$$g_+(t) = c' + ct^{2H} - c(t-1)^{2H} \qquad t \in [1, \infty), \tag{2.24}$$

  for some $c' \in \mathbb{R}$.

- When $H = 1/2$. A particular solution of (2.14) when $t > 1$ is $\log(t) - \log(t-1)$. For the same reason as above, the general solution of (2.14) is thus $d'(\log(t) - \log(t-1))$ where $d' \in \mathbb{R}$. It follows that

$$g_+(t) = d + d'(t \log t - (t-1)\log(t-1)) \qquad t \in [1, \infty), \tag{2.25}$$

  for some $d, d' \in \mathbb{R}$.

Form of $g_+$ on $(0, 1)$ :
Putting $t = 1$ in (2.8) results in

$$(T+1)^{2H} g_+\Big(\frac{T}{T+1}\Big) + T^{2H} g_+\Big(\frac{T+1}{T}\Big) \;=\; g_+(1)[(T+1)^{2H} + T^{2H} - 1].$$

Whence, for $s := T/(T+1) \in (0, 1)$ and using (2.24) one obtains

$$g_+(s) \;=\; -s^{2H} g_+(1/s) + g_+(1)[1 + s^{2H} - (1-s)^{2H}]. \tag{2.26}$$

This gives, for $s \in (0, 1)$,

- when $H \neq 1/2$, $g_+(s) = c' + cs^{2H} - c'(1-s)^{2H}$,
- when $H = 1/2$, $g_+(s) = ds + d'(s \log s - (s-1)\log(s-1))$.

Therefore, relations (2.10) and (2.12) have been proved when $t > 0$.

Form of $g_+$ on $(-\infty, 0)$ : This case follows from (2.8) taking $T = -t$.

Form of $g_-$ : The relations (2.11) and (2.13) are deduced from (2.10) and (2.12) thanks to relation (2.9).

(iv) Follows from the fact that $R$ is the covariance matrix of random vector $(X_1(1)/\sigma_1, \cdots, X_p(1)/\sigma_p)$.

Theorem 2.1 is proved.

# 3 Stochastic integral representation of vfBm

In this section we derive the covariance function of vfBm $X = \{X(t), t \in \mathbb{R}\}$ given by double-sided stochastic integral representation in (1.6). Denote

$$\alpha_{ij}^{++} := \sum_{k=1}^{p} a_{ik}^{+} a_{jk}^{+}, \quad \alpha_{ij}^{--} := \sum_{k=1}^{p} a_{ik}^{-} a_{jk}^{-}, \quad \alpha_{ij}^{+-} := \sum_{k=1}^{p} a_{ik}^{+} a_{jk}^{-}, \quad \alpha_{ij}^{-+} := \sum_{k=1}^{p} a_{ik}^{-} a_{jk}^{+},$$

where $A_+ = \left(a_{ij}^{+}\right), A_- = \left(a_{ij}^{-}\right)$ are the $p \times p$ matrices in (1.6). Clearly,

$$A_+ A_+^* = \left(\alpha_{ij}^{++}\right), \quad A_- A_-^* = \left(\alpha_{ij}^{--}\right), \quad A_+ A_-^* = \left(\alpha_{ij}^{+-}\right), \quad A_- A_+^* = \left(\alpha_{ij}^{-+}\right).$$

Note, each of the processes $X_i = \{X_i(t), t \in \mathbb{R}\}$ in (1.6) is a well-defined fractional Brownian motion with index $H_i \in (0, 1)$; see e.g. Samorodnitsky and Taqqu (1994).

**Proposition 3.1** *The covariance of the process defined in (1.6) satisfies the following properties*

*(i) For any $i = 1, \cdots, p$ the variance of $X_i(1)$ is*

$$\sigma_i^2 = \frac{B(H_i + .5, H_i + .5)}{\sin(H_i \pi)} \left\{\alpha_{ii}^{++} + \alpha_{ii}^{--} - 2\sin(H_i\pi)\alpha_{ii}^{+-}\right\}. \tag{3.1}$$

*(ii) If $H_i + H_j \neq 1$ then for any $s, t \in \mathbb{R}$, the cross-covariance $\mathrm{E}X_i(s)X_j(t)$ of the process in (1.6) is given by (1.4), with*

$$\begin{aligned} \frac{\sigma_i \sigma_j}{2} c_{ij} &:= \frac{B(H_i + .5, H_j + .5)}{\sin((H_i + H_j)\pi)} \\ &\quad \times \left\{\alpha_{ij}^{++} \cos(H_i\pi) + \alpha_{ij}^{--} \cos(H_j\pi) - \alpha_{ij}^{+-} \sin((H_i + H_j)\pi)\right\}. \end{aligned} \tag{3.2}$$

*(iii) If $H_i + H_j = 1$ then for any $s, t \in \mathbb{R}$, the cross-covariance $\mathrm{E}X_i(s)X_j(t)$ of the process in (1.6) is given by (2.2), with*

$$\begin{aligned} \sigma_i \sigma_j d_{ij} &:= B(H_i + .5, H_j + .5) \\ &\quad \times \left\{\frac{\sin(H_i\pi) + \sin(H_j\pi)}{2}(\alpha_{ij}^{++} + \alpha_{ij}^{--}) - \alpha_{ij}^{+-} - \alpha_{ij}^{-+}\right\}, \end{aligned} \tag{3.3}$$

$$\sigma_i \sigma_j f_{ij} := (H_j - H_i)(\alpha_{ij}^{++} - \alpha_{ij}^{--}). \tag{3.4}$$

**Remark 3.2** Let $H_i + H_j \neq 1,\ i,j = 1,\cdots,p$ and let $\tilde{c}_{ij},\ \phi_{ij}$ be defined as in (1.7).

¿From Proposition 3.1 (3.1), (3.2) it follows that the matrix $\tilde{C} = (\tilde{c}_{ij})$ satisfies (1.8). In this context, a natural question arises to find easily verifiable conditions on the matrices $\tilde{C}$ and $H$ such that there exist matrices $A_+, A_-$ satisfying the quadratic matrix equation in (1.8). In other words, for which $\tilde{C}$ and $H$ there exists a vfBm $X$ with cross-covariance as in (1.4)?

While the last question does not seem easy, it becomes much simpler if we restrict the class of vfBm's $X$ in (1.6) to *causal* representations with $A_- = 0$. In this case, equation (1.8) becomes

$$\tilde{C} = \cos(H) A_+ A_+^*.$$

Clearly, the last factorization is possible if and only if the matrix $\cos(H)^{-1}\tilde{C}$ is symmetric and positive definite.

*Proof.* For all $s$, let

$$\begin{aligned} I^+_{ik}(s) &:= \int_{\mathbb{R}} \left( (s-x)_+^{H_i - .5} - (-x)_+^{H_i - .5} \right) W_k(\mathrm{d}x), \\ I^-_{ik}(s) &:= \int_{\mathbb{R}} \left( (s-x)_-^{H_i - .5} - (-x)_-^{H_i - .5} \right) W_k(\mathrm{d}x). \end{aligned}$$

Using the above notation, $X_i(s) = \sum_{k=1}^p \left( a^+_{ik} I^+_{ik}(s) + a^-_{ik} I^-_{ik}(s) \right)$ and

$$\begin{aligned} \mathrm{E} X_i(s) X_j(t) &= \alpha^{++}_{ij} \mathrm{E} I^+_{i1}(s) I^+_{j1}(t) + \alpha^{+-}_{ij} \mathrm{E} I^+_{i1}(s) I^-_{j1}(t) \\ &+ \alpha^{-+}_{ij} \mathrm{E} I^-_{i1}(s) I^+_{j1}(t) + \alpha^{--}_{ij} \mathrm{E} I^-_{i1}(s) I^-_{j1}(t). \end{aligned} \tag{3.5}$$

Let $H_i + H_j \neq 1$. From Stoev and Taqqu (2006, Th. 4.1), taking there $a^+ = 1$, $a^- = 0$, $H(s) = H_i$ and $H(t) = H_j$, we obtain

$$\begin{aligned} \mathrm{E} I^+_{i1}(s) I^+_{j1}(t) &= \psi_H \Big[ \cos\Big( (H_j - H_i)\frac{\pi}{2} - \frac{(H_i + H_j)\pi}{2} \mathrm{sign}(s) \Big) |s|^{H_i + H_j} \\ &+ \cos\Big( (H_j - H_i)\frac{\pi}{2} + \frac{(H_i + H_j)\pi}{2} \mathrm{sign}(t) \Big) |t|^{H_i + H_j} \\ &- \cos\Big( (H_j - H_i)\frac{\pi}{2} - \frac{(H_i + H_j)\pi}{2} \mathrm{sign}(s-t) \Big) |s-t|^{H_i + H_j} \Big], \end{aligned}$$

where

$$\psi_H := \frac{\Gamma(H_i + .5)\Gamma(H_j + .5)\Gamma(2 - H_i - H_j)}{\pi (H_i + H_j)(1 - H_i - H_j)} = \frac{B(H_i + .5, H_j + .5)}{\sin((H_i + H_j)\pi)}.$$

Therefore,

$$\mathrm{E} I^+_{i1}(s) I^+_{j1}(t) = \frac{B(H_i + .5, H_j + .5)}{\sin((H_i + H_j)\pi)} \Big[ b_{ij}(s)|s|^{H_i + H_j} + b_{ji}(t)|t|^{H_i + H_j} - b_{ij}(s-t)|s-t|^{H_i + H_j} \Big],$$

where

$$b_{ij}(s) = \begin{cases} \cos(H_i\pi), & \text{if } s > 0 \\ \cos(H_j\pi), & \text{if } s < 0. \end{cases}$$

Similarly, taking $a^+ = 0$, $a^- = 1$, $H(s) = H_i$ and $H(t) = H_j$ in Stoev and Taqqu (2006, Th. 4.1), we obtain

$$\mathrm{E}I_{i1}^-(s)I_{j1}^-(t) = \frac{B(H_i + .5, H_j + .5)}{\sin((H_i + H_j)\pi)}\Big[b_{ji}(s)|s|^{H_i+H_j} + b_{ij}(t)|t|^{H_i+H_j} - b_{ji}(s-t)|s-t|^{H_i+H_j}\Big].$$

Finally,

$$\begin{aligned}\mathrm{E}I_{i1}^+(s)I_{j1}^-(t) &= \Big(\mathbf{1}_{\{s>t\}}\int_t^s (s-x)^{H_i-.5}(x-t)^{H_j-.5}\mathrm{d}x - \mathbf{1}_{\{s>0\}}\int_0^s (s-x)^{H_i-.5}x^{H_j-.5}\mathrm{d}x \\ &\qquad -\mathbf{1}_{\{t<0\}}\int_t^0 (-x)^{H_i-.5}(x-t)^{H_j-.5}\mathrm{d}x\Big) \\ &= B(H_i+.5, H_j+.5)\Big((s-t)_+^{H_i+H_j} - s_+^{H_i+H_j} - t_-^{H_i+H_j}\Big) \qquad (3.6)\end{aligned}$$

and

$$\mathrm{E}I_{i1}^-(s)I_{j1}^+(t) = B(H_i+.5, H_j+.5)\Big((t-s)_+^{H_i+H_j} - t_+^{H_i+H_j} - s_-^{H_i+H_j}\Big). \qquad (3.7)$$

Substituting these formulas into (3.5) we obtain (3.2) and (3.1).

Next, let $H_i + H_j = 1$. We get similarly from Theorem 4.1 of Stoev and Taqqu (2006)

$$\begin{aligned}\mathrm{E}I_{i1}^+(s)I_{j1}^+(t) &= \frac{1}{\pi}B(H_i+.5, H_j+.5)\Big[\frac{\pi}{2}\sin(H_i\pi)(|s|+|t|-|s-t|) \\ &\qquad - \cos(H_i\pi)(s\log|s| - t\log|t| - (s-t)\log|s-t|)\Big]\end{aligned}$$

and

$$\begin{aligned}\mathrm{E}I_{i1}^-(s)I_{j1}^-(t) &= \frac{1}{\pi}B(H_i+.5, H_j+.5)\Big[\frac{\pi}{2}\sin(H_i\pi)(|s|+|t|-|s-t|) \\ &\qquad + \cos(H_i\pi)(s\log|s| - t\log|t| - (s-t)\log|s-t|)\Big].\end{aligned}$$

Expressions (3.6) and (3.7) remain true when $H_i + H_j = 1$ and they can be rewritten as

$$\mathrm{E}I_{i1}^+(s)I_{j1}^-(t) = \mathrm{E}I_{i1}^-(s)I_{j1}^+(t) = -\frac{1}{2}B(H_i+.5, H_j+.5)(|s|+|t|-|s-t|).$$

Therefore, using (3.5), we obtain (3.3) and (3.4). Proposition 3.1 is proved.

## References


Aczél, J. (1966) *Lectures on functional equations and their applications.* Academic Press, New York.

Aczél, J. and Hosszú, M. (1965) Further uniqueness theorems for functional equations. *Acta. Math. Acad. Sci. Hung.* **16**, 51–55.

Chung, C.-F. (2002) Sample means, sample autocovariances, and linear regression of stationary multivariate long memory processes. *Econometric Th.* **18**, 51–78.

Davidson, J. and de Jong, R.M. (2000) The functional central limit theorem and weak convergence to stochastic integrals. *Econometric Th.* **16**, 643–666.

Davidson, J. and Hashimadze, N. (2008) Alternative frequency and time domain versions of fractional Brownian motion. *Econometric Th.* **24**, 256–293.

Didier, G. and Pipiras, V. (2008) Integral representations of operator fractional Brownian motion. Preprint.

Hudson, W. and Mason, J. (1982) Operator-self-similar processes in a finite-dimensional space. *Trans. Amer. Math. Soc.* **273**, 281–297.

Laha, R.G. and Rohatgi, V.K. (1981) Operator self-similar stochastic processes in $R^d$. *Stochastic Proces. Appl.* **12**, 73–84.

Lavancier, F., Philippe, A. and Surgailis, D. (2009) A two-sample test for comparison of long memory parameters. arXiv:0907.1787v1 [math.ST]

Maejima, M. and Mason, J. (1994) Operator-self-similar stable processes. *Stochastic Proces. Appl.* **54**, 139–163.

Marinucci, D. and Robinson, P.M. (2000) Weak convergence of multivariate fractional processes. *Stochastic Process. Appl.* **86**, 103–120.

Robinson, P.M. (2008) Multiple local Whittle estimation in stationary systems. *Ann. Statist.* **36**, 2508–2530.

Samorodnitsky, G. and Taqqu, M.S. (1994) *Stable Non-Gaussian Random Processes.* Chapman and Hall, New York.

Sato, K. (1991) Self-similar processes with independent increments. *Probab. Th. Rel. F.* **89**, 285–300.

Stoev, S. and Taqqu, M.S. (2006) How rich is the class of multifractional Brownian motions? *Stochastic Process. Appl.* **11**, 200–221.